Addition by Fourier transform

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This corresponds to problem 5.6 in Nielsen & Chuang. The original paper is (Draper 2000). Which quantum circuit can be used to perform the computation

$$|x\rangle \quad \to \quad |x+y \mod 2^n\rangle$$

with $0 \leq x < 2^n$ and a constant **integer** y.

We exploit the general idea

$$x + y = \log\left(\mathrm{e}^x \mathrm{e}^y\right)$$

where the exponentiation is de facto performed by a Fourier trafo and the logarithm by the inverse trafo.

Fourier transforming the state $|x\rangle$ with n bits, leads to the following product representation

$$|x\rangle \ = |x_n x_{n-1} \ldots x_1\rangle \ \to \ \frac{1}{2^n}(|0\rangle + e^{2\pi i 0.x_1}|1\rangle)(|0\rangle + e^{2\pi i 0.x_2 x_1}|1\rangle)\cdots(|0\rangle + e^{2\pi i 0.x_n \ldots x_1}|1\rangle)$$

where we use the notation

$$x = x_1 2^0 + x_2 2^1 + \ldots + x_n 2^{n-1}$$

and

$$0.x_l \ldots x_1 \ \equiv \ \frac{x_l}{2} + \frac{x_{l-1}}{2^2} + \ldots + \frac{x_1}{2^l}\,.$$

Now, we apply a phase shift $R_\theta(\theta)$ to each qubit

$$R_z \ \equiv \ \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\theta) \end{pmatrix}\,.$$

We apply R_θ with $\theta_j = 2\pi y/2^{n-(j-1)}$ to qubit j where $1 \leq j \leq n$. For y we can also write

$$y \ = \ y_1 2^0 + y_2 2^1 + \ldots + y_n 2^{n-1}\,.$$

Thus,

$$\exp(2\pi i y/2^{n-j+1}) = \prod_{k=0}^{n-1} \exp(2\pi i y_{k+1} 2^{j-1-n+k})\,.$$

Since $\exp(2\pi i y_k l) = 1$ for positive integer l, this reduces to (recall $y_k \in \{0,1\}$)

$$\exp(2\pi i y/2^{n-j+1}) = \prod_{k=0}^{n-j} \exp(2\pi i y_{k+1} 2^{j-1-n+k})\,.$$

The nth qubit gets multiplied with $\exp(i\theta_n)$ with $\theta_n = 2\pi y/2^1$. Thus, we need to compute

$$\exp(2\pi i x_1/2)\cdot\exp(2\pi i y_1/2) = \exp(2\pi i(x_1+y_1)/2)\,.$$

Similarly, for the jth qubit one gets

$$\exp(2\pi i(x_1/2^{n-j+1} + x_2/2^{n-j} + \ldots))\cdot\exp(2\pi i(y_1/2^{n-j+1} + y_2/2^{n-j} + \ldots)) = \exp(2\pi i((x_1+y_1)/2^{n-j+1} + (x_2+y_2)/2^{n-j} + \ldots))$$

which implements the addition $\mod n$ operation in this binary fraction.

Now apply the inverse Fourier trafo and it is easy to see that this transforms back to the state $|x + y \mod n\rangle$.

For the practical implementation we first need the phase shift operators, which is up to a phase identical to R_z:

```
Rtheta <- function(bit, theta=0.) {
  return(methods::new("sqgate", bit=as.integer(bit),
                      M=array(as.complex(c(1, 0, 0, exp(1i*theta))),
                              dim=c(2,2)), type="Rt"))
}
```

With this one can write the desired function on state x.

```
addbyqft <- function(x, y) {
  n <- x@nbits
  z <- qsimulatR::qft(x)
  for(j in c(1:n)) {
    z  <- Rtheta(bit=j, theta = 2*pi*y/2^(n-j+1)) * z
  }
  z <- qft(z, inverse=TRUE)
  return(invisible(z))
}
```

Examples

```
x <- qstate(5, basis=as.character(seq(0, 2^5-1)))
x
```

```
   ( 1 )    * 0
```

```
z <- addbyqft(x, 3)
z
```

```
   ( 1 )    * 3
```

```
z <- addbyqft(z, 5)
z
```

```
   ( 1 )    * 8
```

```
z <- addbyqft(z, 30)
z
```

```
   ( 1 )    * 6
```

References

Draper, Thomas G. 2000. "Addition on a Quantum Computer." *arXiv Preprint Quant-Ph/0008033*.